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MEMOIRS

OF THE

COLLEGE OF SCIENCE

KYOTO IMPERIAL UNIVERSITY

VOL. V, No. 6

JULY 1922

KYOTO

Electrolytic Reactions of Naphthalene and its Derivatives. Part III. Electrolytic Oxidation of α-Naphthylamine and ar-Tetrahydro-α-Naphthylamine

By

Kashichi Ono.

(Received March 8, 1922)

(1) Electrolytic Oxidation of α-Naphthylamine.

The electrolytic oxidation of the aromatic compounds containing the amine group has been studied by Goppelsröder and others.[1] They obtained aniline black from aniline and complex colouring matter from other aromatic amines as the chief oxidation products. Sometimes oxidation was observed by them to progress still further so as to change the aromatic amines into the quinone compounds, as in the case of aniline[2] from which p-benzoquinone was produced.

As to the electrolytic oxidation of α-naphthylamine there is apparently no other record in chemical literature than the work of Goppelsröder[3] who is reported to have obtained naphthylamine-violet. So I have taken up the problem and undertaken to study the progress of the oxidation reaction, in the expectation that it would give some interesting result when compared with the oxidation reactions of naphthalene and α-naphthol previously investigated by me[4].

α-Naphthylamine is known to give naphtameïn when chemically oxidized by ferric chloride or silver nitrate,[5] and α-naphthoquinone by

1 Goppelsröder, C. R., **81**, 944; **82**, 331,1199 (1875); Coquillion, ibid., **81**, 408; **82**, 228 (1875); W. Löb, Electro-chemistry of organic compounds, 195, (1906).

2 Elbs, Chem-Ztg., **17**, 210; D.R.P. 172654 (1903).

3 Loc. cit.

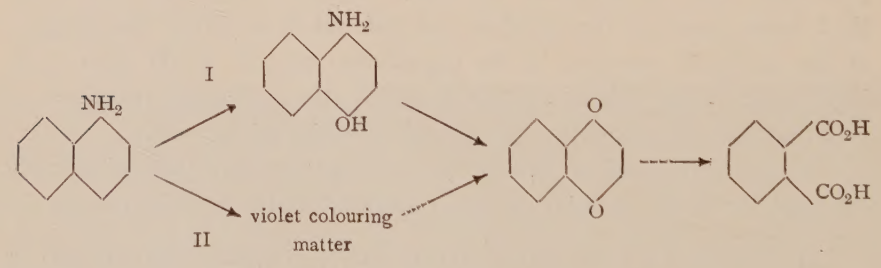
4 These memoirs, **5**, 111,263 (1921).

5 Piria, Lieb. Ann., **78**, 64 (1851); Schiff, ibid., **101**, 92 (1857); **129**, 255 (1865).

potassium bichromate[1] or lead-peroxide[2] in a sulphuric acid solution. According to my experiments described below, electrochemical oxidation gives an anologous result producing a violet colouring matter and α-naphthoquinone as the oxidation product.

As the anode material, platinum, graphite and lead-peroxide were used, of which lead-peroxide gave the best results (Table I). Contrary to the aniline which is solely transformed into aniline black when electrolysed with a platinum electrode[3] α-naphthylamine was found always to produce α-naphthoquinone. This is due to the fact that the violet colouring matter first produced is soluble in acetone used as solvent, while aniline black is insoluble in it. As is evident from table 2, up to about 9 times of the theoretical electric quantity, the increase of the current quantity and that of the production of α-naphthoquinone run parallel. The current yield of the product was observed to become the smaller the greater the current density, the maximum yield having been attained at about 1 amp. per 100 sq. cm. (Table 3). The elevation of the electrolytic temperature does not necessarily favour the yield (Table 4). The suitable concentration of sulphuric acid was 10—30%, and especially was the 10% solution found the best (Table 5). As the oxygencarriers the following substances were found to act effectively :—potassium chromate. chrome-alum, potassium chlorate, and ferric sulphate, etc. (Table 6).

α-Naphthylamine may be assumed to change into α-naphthoquinone by electrochemical oxidation in two different ways, that is, it will first be transformed into amidonaphthol on the one hand and into violet colouring matter on the other, as is shown in the following scheme :



1 P. Monnet, F. Reverdin and E. Noeltig, Ber. D. chem. Ges., **22**, 2036 (1879).
2 K. Ono, not yet published.
3 A. Moser, Die Elektrolytischen Prozesse der org. chem., 68 (1910).

The progress of reaction II was exactly traced in my experiments, so I endeavoured to seize amidonaphthol as the intermediate product and to confirm the fact that reaction I really occurs; but all trials failed. Perhaps it would be instantly further oxidized into the corresponding quinone as soon as it is produced, being very unstable toward the oxidizing action.

(2) Electrolytic Oxidation of ar-Tetrahydro-α-naphthylamine.

It it known that ar-tetrahydro-α-naphthylamine gives a colouring matter like naphtamine by the action of ferric chloride, chromic acid etc.[1] and ar-tetrahydro-α-naphthoquinone[2] by a sulphuric acid solution of potassium bichromate. I have also obtained ar-tetrahydro-α-naphthoquinone by oxidizing it electrolytically. As the anode material, platinum, lead-peroxide and graphite were used, of which lead-peroxide gave the best result (Table 7). The current density at about 1 amp./100 sq. cm. was found the best (Table 9). The most suitable temperature was 18—23° C (Table 10). The concentration of sulphuric acid should be maintained between 10 and 30% (Table 11). As the oxygen-carrier chrome compound and potossium chlorate, etc. acted effectively (Table 12).

EXPERIMENTAL PART.

(1) Electrolytic Oxidation of α-Naphtylamine.

(a) preliminary experiment.

α-Naphthylamine recrystallized from dilute alcohol and melting at 50° was used.

1. The anode liquid consisting of 100 c.c. sulphuric acid (20%), 50 c.c. of acetone and 1 gram of α-naphthylamine, was introduced into a beaker. A porous cell, successively washed with 1% alkali, water, 5% hydrochloric acid and water, was placed in it as the cathode compartment. Sulphuric acid (20%) was taken as the cathode liquid, and a perforated lead plate coated with lead-peroxide about 10·0 × 8·0 cm. served as the anode, and a smaller sheet of lead as the cathode. The beaker was cooled with water to keep the

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1 Bamberger, Ber. D. chem. Ges., **22**, 775 (1889)

2 Bamberger and Lengfeld, ibid. **23**, 1131 (1890)

temperature of the bath always at about 20°. Electrolysis was continued for nearly 2 hours with a curtent of 0·8 ampère.

After electrolysis, the anode liquid was extracted with ether. A violet precipitate insoluble both in an acetone-ether mixture and in water solution, was produced. The precipitate was filtered and washed several times with warm dilute alcohol. The ethreal extract was evaporated, and the residue was subjected to steam distillation. The yellow crystalline distillate, after having been dried, was recrystallized from petroleum ether, whereupon yellow needle crystals, melting at 125° were obtained. The yellow crystals were proved to be α-naphthoquinone by a direct comparison with a pure specimen.

α-Naphthoquinone isolated as above was analysed with the following results:

0·1862 gr. substance gave 0·5153 gr. CO_2 and 0·0665 gr. H_2O

	Found	Cal. ($C_{10}H_6O_2$)
Carbon	75·43	75·93
Hydrogen	3·96	3·83

2. In the second experiment electrolysis was continued for nearly 3 hours with a current of 1·6 ampère under the same conditions as before. The yellow crystals identified as α-naphthoquinone by the colour reaction and by a direct comparison with a pure specimen were obtained on distilling the anode solution with steam. The distillation residue was extracted with ether, and the ethereal extract was washed several times with a saturated sodium carbonate solution. The alkaline liquid thus obtained was neutralized with dilute hydrochloric acid and again extracted with ether. After drying with anhydrous sodium sulphate, the ethereal extract was evaporated and the residue was sublimed. A sublimate, identified as phthalic acid anhydride by the fluorescence reaction, was procured.

(*b*) QUANTITATIVE INVESTIGATION.

Having now confirmed the fact that α-naphthylamine can be oxidized into a violet colouring matter, α-naphthoquinone and phthalic acid, I next attempted to examine the oxidation reaction under various conditions quantitatively. In all experiments I took care to work at a definite current concentration and temperature.

1. Influence of the anode material.

Anode solution: 1·0 gr. α-naphthylamine

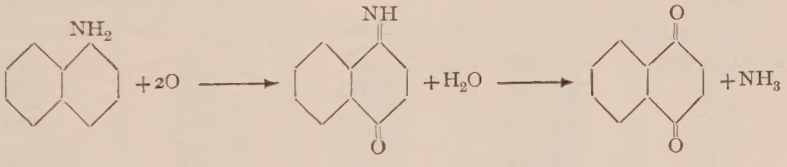
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Cathode solution : H_2SO_4 (20%)
Temperature : 18–23°.

TABLE I.

Anode		Cathode	Current strength	Voltage	Current quantity	α-naphtho-quinone	Current yield of α-naphtho-quinone	Material yield of α-naphtho-quinone
Material	Surface area							
	sq. cm.		amp.	volt	amp. hr.	gr.	%	gr.
Platinum	40·0	Spiral platinum	0·8	3·2–3·4	1·6	0·0097	0·41	0·87
Lead peroxide	80·0	Lead	0·8	2·8–2·9	1·6	0·5643	23·97	51·07
Graphite	45·0	Spiral platinum	0·9	2·6–2·7	1·8	0·0063	0·23	0·57

The anode liquid introduced into a porous cell was constantly agitated by a rapidly revolving stirrer.

After electrolysis the anode liquid was extracted with ether. The solvent was then distilled off and the residue was subjected to steam distillation. *α*-Naphthoquinone obtained thus was collected in a Gooch crucible, dried and weighed. The current yield was calculated on the assumption that oxidation takes place according to the following equation :



2. Influence of current quantity.

Anode solution : 1·0 gr. *α*-naphthylamine
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Anode : Lead peroxide (80·0 sq. cm.)
Cathode solution : H_2SO_4 (20%)
Cathode : Lead
Current density : 0.02 ampère per 1 sq. cm.
Temperature : 18–23°.

TABLE II.

Current quantity	α-naphthoquinone	α-naphthoquinone / Current quantity	Current yield of α-naphthoquinone	Material yield of α-naphthoquinone
amp. hr.	gr.	gr./amp. hr.	%	%
0.8	0·3859	0.4831	32.73	34.92
1.6	0·4490	0.2806	19.04	40.63
3.2	0·5169	0.1615	10.95	46.79
4.8	0·5904	0.1230	8.34	53.43
6.4	0·6591	0.1029	6.98	59.65
8.0	0·6110	0.0763	5.18	55.29

3. Influence of current density.

Anode solution : 1·0 gr. α-naphthylamine
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Anode : Lead peroxide (80 sq. cm.)
Cathode solution : H_2SO_4 (20%)
Cathode : Lead
Current quantity : 1·6 ampère hours
Temperature : 18–23°.

TABLE III.

Current density	Voltage	α-naphthoquinone	Current yield of α-naphthoquinone	Material yield of α-naphthoquinone
amp./100 cm^2.	volt	gr.	%	%
3	3.4—3.6	0.3355	14.22	30.36
2	3.2—3.3	0.4490	19.04	40.63
1	2.8—2.9	0.5643	23.97	51.07
0.5	2.5—2.7	0.5027	21.31	45.49

4. Influence of temperature.

Anode solution : 1.0 gr. α-naphthylamine
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Anode : Lead peroxide (80·0 sq. cm.)
Cathode solution : H_2SO_4 (20%)
Current strength : 0·8 ampère
Current quantity · 1·6 ampère hours.

TABLE IV.

Temperature	Voltage	α-naphthoquinone	Current yield of α-naphthoquinone	Material yield of α-naphthoquinone
°C.	volt	gr.	%	%
18—23	2·8—2·9	0·5643	23·97	51·07
40—50	3·1—3·3	0·3264	13·84	29·54

5. Influence of the concentration of sulphuric acid.

Anode solution: 1·0 gr. *α*-naphthylamine
50 c.c. acetone
100 c.c. H_2SO_4

Anode: Lead peroxide (80·0 sq. cm.)

Cathode solution: H_2SO_4 (the same concentration as in the anode solution)

Cathode: Lead

Current strength: 0.8 ampère

Current quantity: 1.6 ampère hours

Temperature: 18–23°

TABLE V.

Concentration of sulphuric acid	Voltage	α-naphthoquinone	Current yield of α-naphthoquinone	Material yield of α-naphthoquinone
%	volt	gr.	%	%
10	2·7—2·8	0·5918	25·09	53·56
20	2·8—2·9	0·5643	23·97	51·07
30	3·1—3·2	0·5040	21·37	45·61
40	3·3—3·5	0·3892	18·79	35·22

6. Influence of the oxygen carrier.

Anode solution: 1·0 gr. *α*-naphthylamine
50 c.c. acetone
100 c.c. H_2SO_4 (10%)
0·5 gr. oxygen carrier

Anode: Lead peroxide (80.0 sq. cm.)

Cathode solution: H_2SO_4 (10%)

Cathode : Lead
Current strength : 0.8 ampère
Current quantity : 0·8 ampère hour
Temperature : 18–23°

TABLE VI.

Oxygen carrier	Voltage	α-naphtho-quinone	Current yield of α-naphtho-quinone	Material yield of α-naphtho-quinone
	volt	gr.	%	%
Chrome alum	2·6—2·7	0·5614	43·31	50·85
Potassium chromate	2·4—2·6	0·6828	57·55	61·84
Potassium ferrocyanide	2·6—2·8	0·5193	45·77	47·02
Ferric sulphate	2·8—2·9	0·5263	44·36	47·67
Potassium chlorate	2·7—2·9	0·5900	49·73	53·44

(*c*) OXIDATION OF A VIOLET COLOURING MATTER.

0·2 gram of a violet colouring matter was dissolved in a mixture of 50 c.c. of acetone and 20 c.c. of 10% sulphuric acid. A perforated lead plate coated with lead peroxide, about 40 sq. cm. served as the anode, and electrolysis was continued for 4 hours with a current of 0.4 ampère.

After electrolysis was over the anode liquid was extracted with ether. The ethereal extract was then evaporated, and the residue was subjected to steam distillation. The distillation product (about 0·108 gram) was identified as *α*-naphthoquinone by the colour reactions. It was recrystallized from petroleum ether and was found to melt at 125.° By comparing with a pure specimen, it was confirmed to be *α*-naphthoquinone. Phthalic acid was detected in the residue of the steam distillation.

(*d*) OXIDATION OF 1–4–AMIDONAPHTHOL.

One gram of 1–4–amidonaphthol was dissolved in a mixture of 50 c.c. of acetone and 100 c.c. of 10% sulphuric acid. A perforated lead plate coated with lead peroxide, about 80·0 sq. cm. served as the anode, and electrolysis was continued for 1·5 hours with a current of 1·0 ampère.

After electrolysis the anode liquid was treated in an anologous

manner as in the previous experiments, the substance identified as α-naphthoquinone by the colour reactions was isolated. On recrystallizing it from petroleum ether, 0·463 gram of the quinone melting at 125° was obtained.

α-Naphthoquinone isolated as above was analyzed with the following results:

0·1764 gr. substance gave 0·4888 gr. CO_2 and 0·0639 gr. H_2O.

	Found	Cal. ($C_{10}H_6O_2$)
Carbon	75·57	75·93
Hydrogen	4·02	3·83

(2) Electrolytic Oxidation of ar-Tetrahydro-α-naphthylamine.

ar-Tetrahydro-α-naphthlamine sulphate used in my experiments prepared by Eug. Bamberger and F. Lengfeld's[1] method.

0·6397 gr. of the air-dried substance gave 0·0168 gr. water on sulphuric acid.

	Found	Cal. $(C_{10}H_{11}NH_2)_2H_2SO_4+\frac{1}{2}H_2O$
H_2O	2·62	2·24

0·1405 gr. of the anhydrous substance gave 0·0826 gr. $BaSO_4$.

	Found	Cal. $(C_{10}H_{11}NH_2)_2H_2SO_4$
H_2SO_4	24·68	25·01

(a) Preliminary Experiment.

1. The anode liquid, consisting of 100 c.c. of sulphuric acid (20%), 50 c.c. of acetone and 1·0 gram of ar-tetrahydro-α-naphthylamine sulphate, was introduced into a beaker. A porous cell was placed in it as the cathode compartment and sulphuric acid (20%) was poured into the beaker as the cathode liquid. As the anode a perforated lead plate coated with lead-peroxide, about 10·0 × 8·0 cm. was used. The bath was cooled with running water to keep the temperature of it always at about 20°. Electrolysis was continued for nearly one hour with a current of 1·6 ampère.

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1 Ber. D. chem. Ges., **23**, 1131 (1890).

On treating the anode solution in the usual way yellow needle crystals, melting at 55·5°, were obtained.

The yellow needle crystals were analyzed wirh the following results:

1. 0·1483 gr. substance gave 0·3993 gr. CO_2 and 0·0914 gr. H_2O.
2. 0·1326 gr. substance gave 0·3585 gr. CO_2 and 0·0775 gr. H_2O.

	1.	2.	Cal. ($C_{10}H_{10}O_2$)
Carbon	73·43	73·73	74·07
Hydrogen	6·84	6·54	6·17

From the result of this analysis there is no doubt that the yellow needle crystalline substance above obtained is ar-tetrahydro-*α*-naphthoquinone. By comparing it with a specimen prepared by Bamberger and Lengfeld's[1] method this was confirmed. A dark violet substance was obtained as the by-product but it was not further examined.

2. In the second experiment electrolysis was continued for nearly 3 hours with a current of 1·6 ampère. The yellow crystals, identified as ar-tetrahydro-*α*-naphthoquinone by a direct comparison with a pure specimen, were produced on distilling the anode solution with steam. The distillation residue was extracted with ether. The ethereal liquid was shaken up several times with a saturated sodium carbonate solution so as to dissolve any acid produced. The sodium carbonate solution was then neutralised with dilute sulphuric acid and again extracted with ether. After drying with anhydrous sodium sulphate, the ethereal extract was evaporated to a small volume. After standing it for a few days in the vacuum desiccator, a very small quantity of colourless needles melting at 149–151° was found to crystallize out.

(*b*) QUANTITATIVE INVESTIGATION.

In order to study the above oxidation reaction quantitatively the following experiments were carried out.

1. Influence of the anode material.

Anode solution: 1·0 sulphate
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Cathode solution: H_2SO_4 (20%)
Temperature: 18–23°.

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[1] Loc. cit.

TABLE VII.

Anode		Cathode	Current strength	Voltage	Current quantity	ar-tetra-hydro-α-naphtho-quinone	Yield
Material	Surface area						
	sq. cm.		amp.	volt	amp. hr.	gr.	%
Platinum	40·0	Spiral platinum	0·8	3·1—3·2	1·6	0·0081	2·18
Lead peroxide	80·0	Lead	0·8	2·7—2·8	1·6	0·1833	50·01
Graphite	45·0	Spiral platinum	0·9	2·5—2·7	1·8	very little	—

2. Influence of current quantity.

Anode solution: 1·0 gr. sulphate
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Anode: Lead peroxide (80·0 sq. cm.)
Cathode solution: H_2SO_4 (20%)
Cathode: Lead
Current density: 0·02 ampère per 1 sq. cm.
Temperature: 18–23°.

TABLE VIII.

Current quantity	ar-tetrahydro-α-naphthoquinone	Quinone / Current quantity	Yield
amp. hr.	gr.	gr./amp. hr.	%
1·6	0·1083	0·0676	29·54
2·4	0·1532	0·0638	31·80
3·2	0·1944	0·0607	53·04
4·0	0·1872	0·0468	51·07
4·8	0·1380	0·0287	37·65

3. Influence of current density.

Anode solution: 1·0 gr. sulphate
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Anode: Lead peroxide (80·0 sq. cm.)
Cathode solution: H_2SO_4 (20%)
Cathode: Lead
Current quantity: 1·6 ampère
Temperature: 18–23°.

Table IX.

Current density	Voltage	ar-tetrahyro-α-naphthoquinone	Yield
amp./100 cm².	volt	gr.	%
3	3·1—3·2	0·0780	21·28
2	2·8—2·9	0·1083	29·54
1	2·7– 2·8	0 1591	43·41

4. Influence of temperature.

Anode solution: 1·0 gr. sulphate
50 c.c. acetone
100 c.c. H_2SO_4 (20%)
Anode: Lead peroxide (80·0 sq. cm.)
Cathode solution: H_2SO_4 (20%)
Cathode: Lead
Current strength: 0·8 ampère
Current quantity: 1·6 ampère hours.

Table X.

Temperature	Voltage	ar-tetrahydro-α-naphthoquinone	Yield
°C.	volt	gr.	%
7—10	2·6—2·7	0·1291	35·22
18—23	2·7—2·8	0·1591	43·41
49—50	3·0—3·1	0·0526	14·35

5. Influence of the concentration of sulphuric acid.

Anode solution: 1·0 gr. sulphate
50 c.c. acetone
100 c.c. H_2SO_4
Anode: Lead peroxide (80·0 sq. cm.)
Cathode solution: H_2SO_4 (The same concentration as in the anode solution)
Cathode: Lead
Current strength: 0·8 ampère
Current quantity: 1·6 ampère hours
Temperature: 18–23°.

Table XI.

Concentration of sulphuric acid	Voltage	ar-tetrahydro-α-naphthoquinone	Yield
%	volt	gr.	%
10	2·6—2·8	0·1833	50·01
20	2·7—2·8	0·1591	43·41
30	3·0—3·1	0·1205	32·87
40	3·3—3·5	0·0829	22·61

6. Influence of the oxygen carrier.

Anode solution : 1·0 gr. sulphate
50 c.c. acetone
100 c.c. H_2SO_4 (10%)
0·3 gr. oxygen carrier
Anode : Lead peroxide (80·0 sq. cm.)
Cathode solution : H_2SO_4 (10%)
Cathode : Lead
Current strength : 0·8 ampère
Current quantity : 0·8 ampère hour
Temperature : 18–20.

Table XII.

Oxygen carrier	Voltage	ar-tetrahydro-α naphthoquinone	Yield
	volt	gr.	%
Chrome alum	2·6—2·7	0·1026	27·99
Potassium chromate	2·5—2·7	0·1138	31·05
Potassium ferrocyanide	2·9—3·1	0·0993	27·09
Potassium chlorate	3·0—3·1	0·1294	35·30

The author takes pleasure in heartily thanking Prof. M. Matsui for suggesting this research, and for his valuable guidance and kindly interest throughout the work.

Electrolytic Oxidation of Alcohols. I. Isoamylalcohol.

By

Shunzo Koizumi.

(Received March 11, 1922.)

Isovaleric acid is usually prepared by oxidising isoamylalcohol with potassium-bichromate in a sulphuric acid solution. The yield, however, is not quite satisfactory as aldehyde, ester and other substances are formed together, and their separation from the acid requires rather a tedious process.

As to the electrolytic oxidation of isoamylalcohol there has not been much investigation, the only record we could find in chemical literature being the work of Elbs and Brunner.[1] By using a dilute sulphuric acid solution of the alcohol as the anode solution and lead coated with its peroxide as the anode they obtained isovaleric acid as the main oxidation product (about 80% current yield) with a small quantity of carbonic acid.

On the quantitative side of the electrolytic oxidation and the influence of the external conditions upon the reaction, as there were found no further details in their report the author has taken up the problem to investigate it quantitatively, and to find out, if possible, the best method for the preparation of isovaleric acid.

EXPERIMENTAL PART.

The isoamylalcohol used in these experiments was the commercial amylalcohol (b.p. 128–132°) obtained from Boake, Roberts Co. It was purified by repeated fractionations and dehydrated; the physical pro-

1 Zeits. Elektrochem., 6, 608 (1900).

perties of the substance thus purified were as follows: b.p., 130°–132° sp. gr., 0·811. Evidently the portion thus fractionated might still have contained a small quantity of active amylalcohol, but it did no harm to our experiments in the present investigation.

I. Preliminary Experiment.

1. The anode solution prepared by dissolving 22 grams of isoamylalcohol in 1 liter of 10% sulphuric acid was poured into a tall beaker, in which a porous cell was immersed to form the cathode compartment, which contained 10% sulphuric acid as the cathode solution. A perforated lead plate of 200 sq. cm. coated with lead peroxide was served as the anode, while a piece of nickel net was used as the cathode. Electrolysis was continued for about 9 hours with a current of 3 ampères, keeping the temperature of the bath always at nearly 18° by cooling it with running water.

After electrolysis the anode solution was carefully neutralised with sodium hydroxide and then it was distilled to half its volume. The distillate was tested for the presence of aldehyde with "the Fuchsin aldehyde reagent," when pink coloration scarcely occurred even after half an hour. The residual solution was evaporated to dryness on a water bath, the residue was redissolved in some water and was made acidic with dilute sulphuric acid. When the acid solution was subjected to steam distillation a colourless oil was distilled together with water. The oily layer was separated with a separating funnel and the water solution was saturated with sodium sulphate added with some potassium bisulphate and repeatedly extracted with ether. On evaporating the ether, a second portion of oil was obtained which was mixed together with the main portion and distilled, whereupon about 15 grams of an oil distilling at 165–176° was produced.

As this oil is a mixture of the hydrated acid $C_5H_{10}O \cdot H_2O$ (b.p., 165°) and the anhydrous one $C_5H_{10}O_2$ (b.p. 173–176°), it was dehydrated with anhydrous sodium sulphate and then redistilled, when about 12 grams of the anhydrous acid was obtained. From its physical properties (b.p. 173–177°, sp. gr. 0·938) and other points it was confirmed to be isovaleric acid.

2. In the second experiment acetone was added to the anode solution in order to make the concentration of amylalcohol greater, the other conditions having been maintained the same as in the first experiment.

Anode solution: 300 c.c. H_2SO_4 (10%).
200 c.c. Acetone.
50 gr. Isoamylalcohol.

After electrolysing for about 15 hours with a current of 3 ampères a very small quantity of isovaleric acid was formed.

3. A sodium hydroxide solution was then tried under the following conditions:

Anode:	Nickel net (400 sq. cm.)
Anode Solution:	1000 c.c. NaOH (5%)
	50 gr. isoamylalcohol.
Cathode:	Nickel net.
Cathode Solution:	NaOH (5%)
Temperature:	25–27°

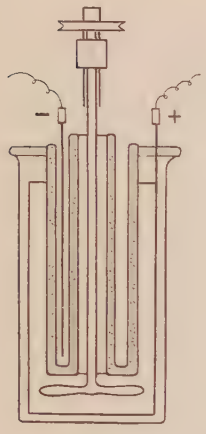

Fig. 1.

In this electrolysis a double-tubed porous cell as in the annexed figure (Fig. 1.) was used, and the anode solution in the beaker was agitated with a stirrer which passed through the inner porcelain tube.

After 12 hours' electrolysis with a current of 8 ampères, the anode solution was neutralised with dilute sulphuric acid and distilled. The distillate showed a distinct reaction for aldehyde. It was therefore extracted repeatedly with ether, and on evaporation of the ether, a small quantity of isovaleric aldehyde was obtained which distilled at about 92–93°.

The valeric acid was isolated in the same manner as in the previous experiment, and the amount was found to be about 10 grams.

4. The electrolysis was repeated under exactly the same conditions as in the 3rd experiment, with merely the addition of 30 grams of potassium bichromate to the anode solution. A current of 6 ampères was passed for 10 hours, and a small amount of aldehyde and about 6 grams of valeric acid were obtained.

5. Substituting potassium permanganate for potassium bichromate the yield of valeric acid was found to increase more than 15%.

II. Quantitative Investigation.

The formation of isovaleric aldehyde and isovaleric acid by the electrolytic oxidation of isoamylalcohol having now been confirmed, the

author went on to investigate quantitatively the procedure of the electrolysis under various conditions.

A. Experiment with the open bath.

6. Influence of anode material.

Anode solution: 400 c.c. H_2SO_4 (10%)
isoamylalcohol 10 gr.
Cathode solution: H_2SO_4 (10%)
Temperature: 17–18°

After the electrolysis the anode solution was neutralised with a caustic soda solution, and it was distilled till about 250 c.c. distilled over. The yield of the aldehyde was calculated by titrating the distillate according to the method described below:

The volumetric estimation of isovaleric aldehyde was performed according to Ripper's or Rocques' method.[1] First, to prepare a sulphurous acid solution (about N/10), 12·5 gr. of pure anhydrous sodium sulphite were dissolved in 400 c.c. of water, 100 c.c. of normal sulphuric acid were added to it and the mixture was diluted to 1 liter with 96% alcohol, and after standing over-night, the sodium sulphate which crystallised out was separated by filtration and the filtrate kept in a well-stoppered bottle. In one of the two long-necked flasks of about 100 c.c. capacity, 25 c.c. of the aldehyde solution was mixed with 50 c.c. of the sulphurous acid solution, while in the other flask the same amount of sulphurous acid solution (50 c.c.) was taken, and both were well stoppered. After two days, each solution was titrated with N/10 iodine solution, using starch as the indicator. The quantity of isovaleric aldehyde was calculated as follows:

$$A=\frac{(a-b)\times\frac{J}{10\times 1000}\times\frac{M}{10}}{J}=(a-b)\times 0{\cdot}004305$$

where A=grams of isovaleric aldehyde in 25 c.c. of the solution; a=titration value of the iodine solution in c.c. for the sulphurous acid solution; b=titration value of the iodine solution in c.c. for the sulphurous acid solution mixed with the aldehyde solution; M=molecular weight of isovaleric aldehyde.

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[1] M. 21, 1079 (1900)—Ann. 341,163 (1905); Chem. C. 69 (1898) II, 1037–70 (1899) I, 65.

In order to determine the valeric acid formed, the distillation residue was acidified and subjected to steam distillation. Some water was added to the distillate so as to make it just 1 liter, and 20 c.c. of it was taken out and titrated with N/10 sodium hydroxide solution, using phenolphthalein as the indicator.

Table I.

Anode	Cathode	Current density	Electric quantity (amp./h.)	Aldehyde (gr.)	Acid (gr.)	Current efficiency (%)
platinum	platinum spiral	1·39	12·17	0·092	3·01	26·40
lead peroxide	nickel net	1·39	12·18	—	5·84	50·43
nickel net	nickel net	1·39	12·20	—	1·20	—

7. Influence of current density.

Anode: Lead peroxide (216 sq. cm.)
Anode solution: 400 c.c. H_2SO_4 (10%)
10 gr. amylalcohol.
Cathode: Nickel net.
Cathode solution: H_2SO_4 (10%)
Temperature: 17–18°.

Table II.

Current strength (amp)	Current density (amp./$_{100}$sq. cm.)	Electric quantity (amp./h.)	Aldehyde	Acid (gr.)	Current efficiency (%)	Material yield (%)
2	0·926	12·18	trace	5·86	50·60	65·20
3	1·391	12·18	—	5·84	50·43	65·80
4	1·85	12·20	—	5·67	48·97	
solution stirred 4	1·85	12·20	—	6·63	57·02	74·01

8. Influence of temperature.

Anode: Lead peroxide (216 sq. cm.)
Anode solution: the same as in exp. 7.
Cathode: nickel net.
Cathode solution: H_2SO_4 (10%)
Current strength: 3 ampères.
Electrlc quantity: 12·2 amp/hour.

Table III.

Temperature	Aldehyde	Valeric acid (gr.)	Current efficiency (%)	Material yield (%)
17–19°	—	5·84	50·43	65·80
40–50°	—	3·79	32·73	42·70

9. Influence of current density in alkaline solution.

Anode : nickel net (228 sq. cm.)
Anode solution : 400 c.c. NaOH (5%)
10 gr. amylalcohol.
Cathode : nickel net.
Cathode solution : NaOH (5%)
Temperature : 18–20°.

After electrolysis, the anode solution was neutralised with dilute sulphuric acid, and each of the aldehyde and the acid was titrated as in the previous experiment.

Table IV.

Current strength (amp.)	Current density (amp./$_{100}$sq. cm.)	Electric quantity (amp./h.)	Aldehyde (gr.)	Acid (gr.)	Current efficiency (%)
5	2·19	15·0	0·097	1·053	7·76
4	1·75	15·0	0·142	1·258	9·38
3	1·33	15·0	0·153	1·784	13·10
2	0·88	15·0	0·213	2·297	16·90
1	0·44	15·0	0·234	4·202	30·34

As is clearly shown in the above table that the current efficiency in an alkaline solution is far worse than that in an acid solution, and that the formation of acid is always accompanied by that of aldehyde, we may conclude that the alkaline solution is not suitable for the electrolytic preparation of isovaleric acid.

B. Experiment with the closed bath provided with an air-tight stirrer.

Having now confirmed that the sulphuric acid solution was most suitable for preparing isovaleric acid by electrolytic oxidation of

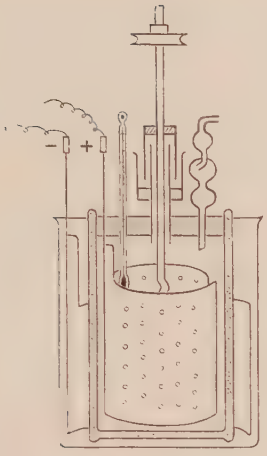


Fig. 2.

isoamylalcohol, the author next attempted to discover with what current efficiency the oxidation proceeds under various conditions, and how the oxidation reaction is influenced by the addition of oxygen carriers. For this purpose the oxygen liberated from the electrolytic bath and from a voltameter was measured using the apparatus shown in figures 2 and 3.

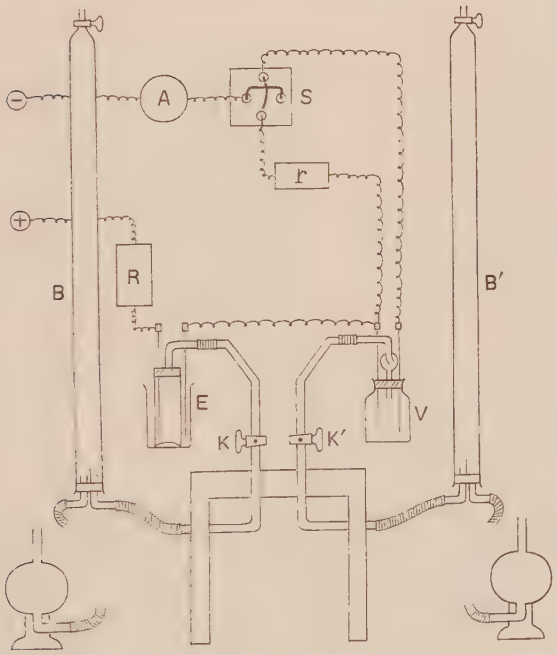


Fig. 3.

The anode solution was placed in an earthenware cylinder which was tightly stoppered with a rubber stopper provided with a thermometer, a gas delivery tube and a mercury gas-tight stirrer. The apparatus was fitted up as is shown in Fig. 3. 40 volts terminals were

used as the current source and the current strength was adjusted by means of a resistance *R*. *V* is a detonating gas coulombmetes, *B*, *B'* are gas-measuring burettes of 100 c.c. capacity graduated every 0·1 c.c., *S* is a two-way-switch which served to connect the bath with the voltameter in series on the one hand, and with a rheostat (*r*) whose resistance was equal to that of the voltameter on the other, *K*. *K'* are three-way-cocks.

The oxygen liberated from the bath and from the coulombmeter every 2 minutes was observed. Thus, knowing the volume of oxygen in c.c. used for the oxidation during every 1 minute period, and taking them on the ordinates and the corresponding time from the beginning of reaction on the abscissa, the following diagrams were traced. The oxidation efficiency was calculated as the ratio of oxygen used up for oxidation to that of oxygen liberated from the voltameter. The current efficiency was calculated from the amount of isovaleric acid produced.

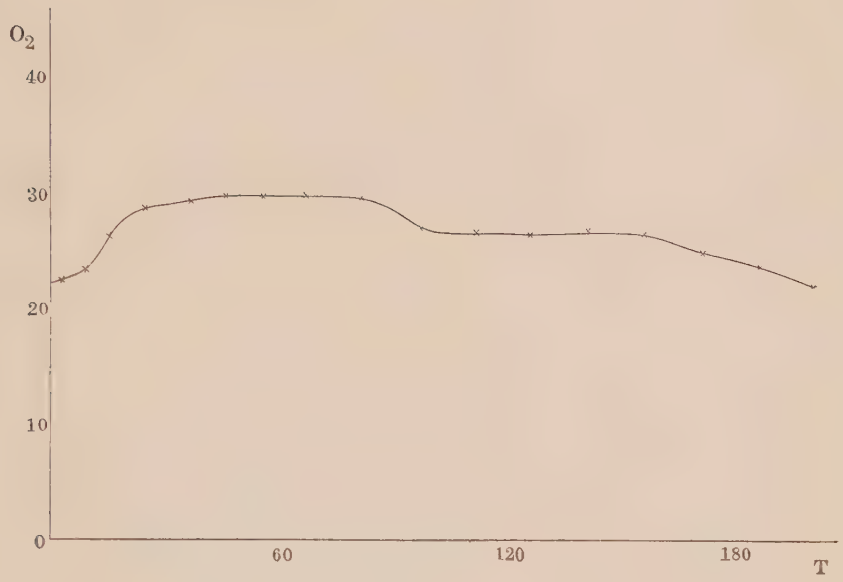
(*a*) Influence of cathode material.

10. Anode: lead peroxide (216 sq. cm.)
Anode solution: 400 c.c. H_2SO_4 (10%)
10 gr. amylalcohol.
Cathode: nickel net.
Cathode solution: H_2SO_4 (10%)
Current strength: 4 ampères.
Temperatures: 18–20°

Oxidation efficiency: 84·40% (3 hours) 83·10% (3.h20m)
Current efficiency: 57·80% 56·90%
Valeric acid formed: 7·19 gr.

Table V.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	22·6	55	30·0	140	27·7
9	23·6	65	30·3	155	27·4
15	26·2	80	30·3	170	25·3
25	29·0	95	27·6	185	24·3
35	29·5	110	27·2	200	22·3
45	30 1	125	27·2		



11. Cathode: lead plate.

The other conditions being the same as in exp. 10.

Oxidation efficiency: 86·76% ($3^{h.}$) 68·10% ($4^{h.}$ $30^{m.}$)

Current efficiency: 55·80% . 43·80%

Valeric acid formed: 7·51 gr.

Table VI.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	16·9	70	28·8	195	23·8
8	25·1	85	28·1	219	22·7
15	25·9	100	28·5	230	23·0
22	29·2	115	27·4	240	23·0
30	30·7	130	27·6	260	23·2
40	30·5	150	28·7	270	22·0
50	29·5	165	27·8	275	20·6
60	29·0	180	27·8		

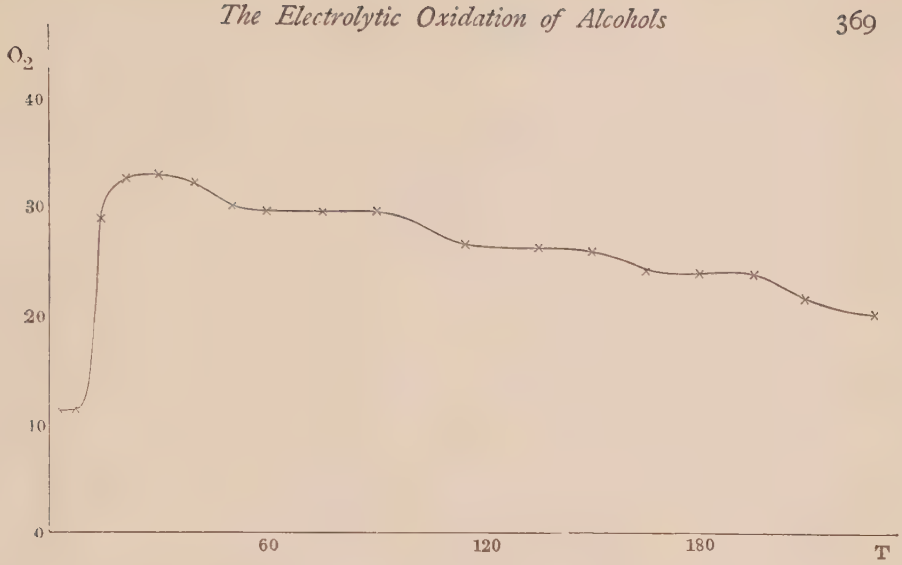
(*b*) Influence of the concentration of sulphuric acid.

12. Anode: lead peroxide (216 sq. cm.)
Anode solution: 400 c.c. H_2SO_4 (5%)
10 gr. isoamylalcohol.
Cathode: lead plate.
Cathode solution: H_2SO_4 (5%)
Temperature: 18–20°
Current strength: 4 amps.

Oxidation efficiency: 84·65% (3^h.) 81·10% (3^h. 50^m.)
Current efficiency: 51·51% 49·40%
Valeric acid formed: 7·20 gr.

Table VII.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	11·4	50	30·1	150	26·1
7	11·5	60	29·8	165	24·4
14	29·0	75	29·8	180	24·1
21	32·8	90	29·9	195	24·1
30	33·0	115	26·8	210	21·8
40	32·4	135	26·6	230	20·4



13. Anode solution: 400 c.c. H_2SO_4 (10%)
10 gr. isoamylalcohol.
Cathode solution: H_2SO_4 (10%)
Other conditions: as in exp. 12.

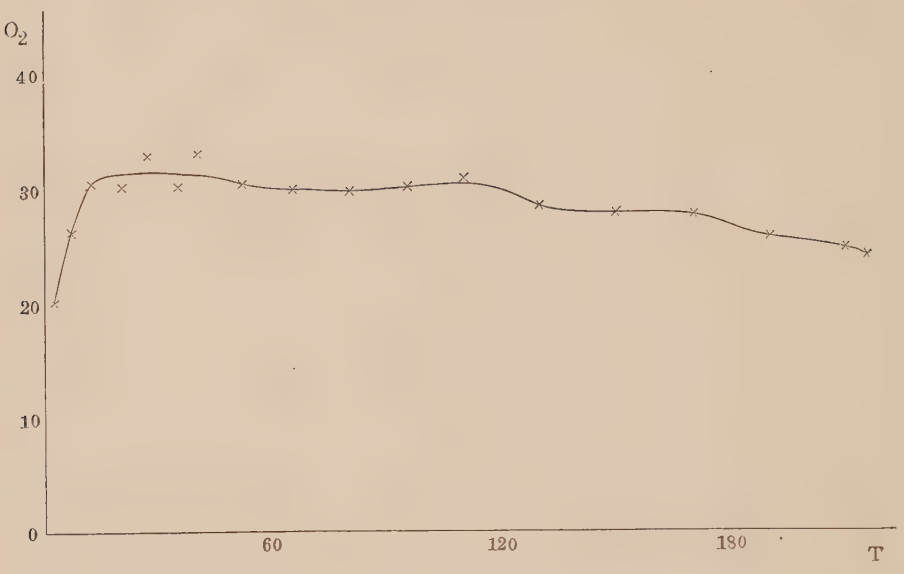
Oxidation efficiency: 86·86% (3^h.) 68·10% (4^h. 30^m.)
Current efficiency: 55·80% 43·80%
Valeric acid formed: 7·51 gr.
See Table VI.

14. Anode solution; 400 c.c. H_2SO_4 (20%)
10 gr. isoamylalcohol.
Cathode solution: H_2SO_4 (20%)
Other conditions: as in exp. 12.

Oxidation efficiency: 88·63% (3^h.) 86·72 (3^h. 35^m.)
Current efficiency: 58·77% 57·45%
Valeric acid formed: 7·80 gr.

Table VIII.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	20·2	40	33·1	130	28·5
7	26·3	52	30·6	150	27·9
12	30·5	65	30·1	170	27·7
20	30·2	80	29·8	190	25·9
27	33·0	95	30·3	210	24·9
35	30·3	110	30·8	215	24·0

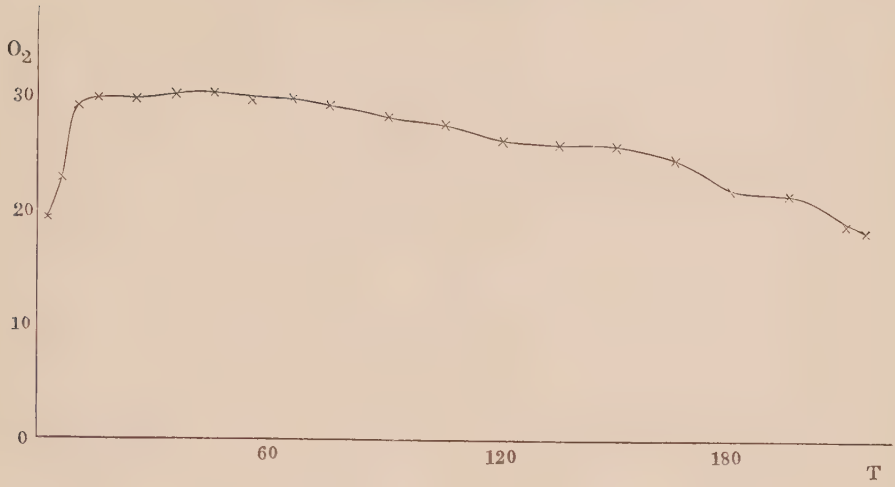


15. Anode solution: 400 c.c. H_2SO_4 (30%)
10 gr. isoamylalcohol.
Cathode solution: H_2SO_4 (30%)
Other conditions: as in exp. 12.

Oxidation efficiency: 83·10% ($3^{h.}$) 79·84% ($3^{h.}$ $35^{m.}$)
Current efficiency: 55·10% 52·91%
Valeric acid formed: 7·22 gr.

Table IX.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	19·3	55	29·9	150	25·9
6	22·7	65	30·0	165	24·8
10	29·1	75	29·4	180	22·0
15	29·9	90	28·4	195	21·7
25	29·8	105	27·8	210	19·2
35	30·2	120	26·4	215	18·5
45	30·3	135	26·0		



(*c*) Influence of current density.

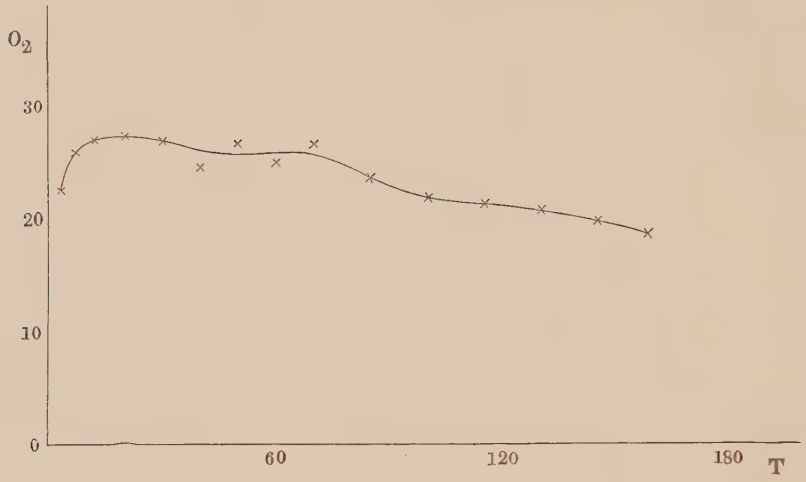
As is evident from the results of previous experiments with the open bath, the current efficiency increases with the decrease of current density. But if the current strength becomes smaller than 4 ampères, oxidation is not complete and some aldehyde is formed. To know what result will be obtained when current strength is increased more than 4 ampères the following experiment was carried out.

16. Other conditions having been maintained the same as in exp. 10. the current of 5 ampères was passed.

Oxidation efficiency: 77·38% ($2^{h.}$ $24^{m.}$) 75·82% ($2^{h.}$ $38^{m.}$)
Current efficiency: 55·09% 53·99%
Valeric acid formed: 6·76 gr.

Table X.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
3	22·4	40	24·4	100	21·8
7	24·7	50	26·6	115	21·1
12	26·9	60	24·9	130	20·6
20	27·1	70	26·5	145	19·5
30	26·7	85	23·5	158	18·4



Both efficiencies are smaller when compared to those obtained by using a current of 4 ampères:

Current density	Oxidation efficiency (12 A.H.)	Current efficiency (12 A.H.)
5 amp.	77·38%	55·09%
4 amp.	86·76%	55·80%

(*d*) Influence of oxygen-carrier.

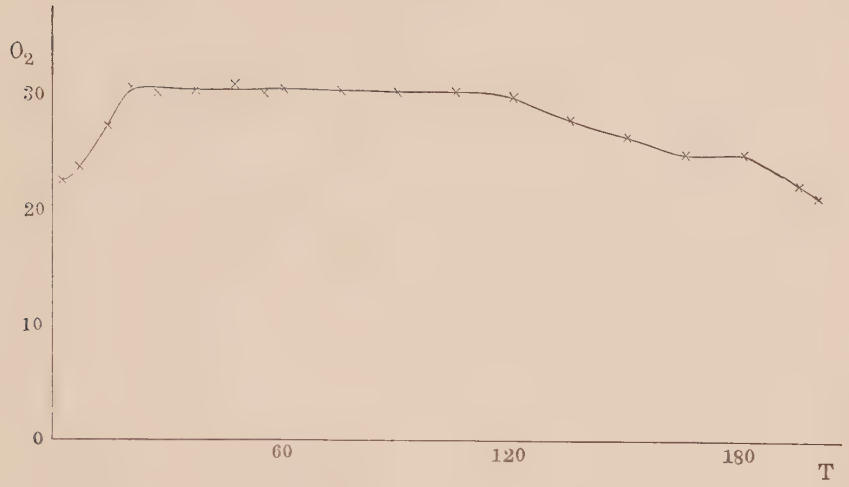
17. Anode solution: 400 c.c. H_2SO_4 (10%)
0·5 gr. Ce_2SO_4
10 gr. isoamylalcohol.

Other conditions as in exp. 10.

Oxidation efficiency: 87·42% ($3^{h.}$) 85·69% ($3^{h.}\ 20^{m.}$)
Current efficiency: 59·89% 58·70%
Valeric acid formed: 7·44 gr.

Table XI.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	22·6	55	30·5	150	26·9
7	23·7	60	30·9	165	25·9
14	27·4	75	30·6	180	25·2
20	30·7	90	30·6	195	22·4
27	30·3	105	30·5	200	21·4
37	30·6	120	30·1		
47	31·1	135	28·3		

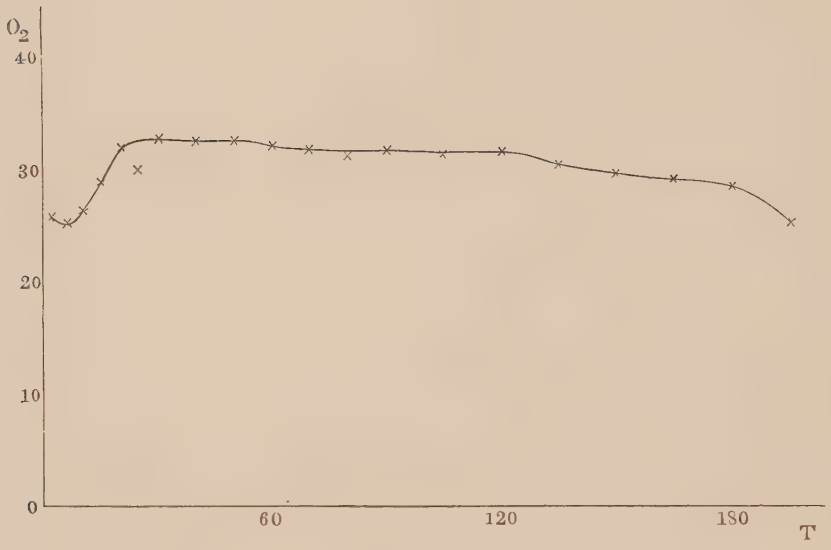


18. 0·5 gr. vanadium pentoxide was used instead of cerium sulphate.

Oxidation efficiency: 92·26% ($3^{h.}$) 91·45% ($3^{h.}$ $15^{m.}$)
Current efficiency: 62·70% 62·15%
Valeric acid formed: 7·69 gr.

Table XII.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	25·9	40	32·1	120	31·4
6	25·1	50	32·3	135	30·1
10	26·2	60	31·8	150	29·4
15	28·8	70	31·5	165	28·9
20	31·8	80	31·0	180	28·2
25	29·9	90	31·4	195	24·9
30	32·5	105	31·1		

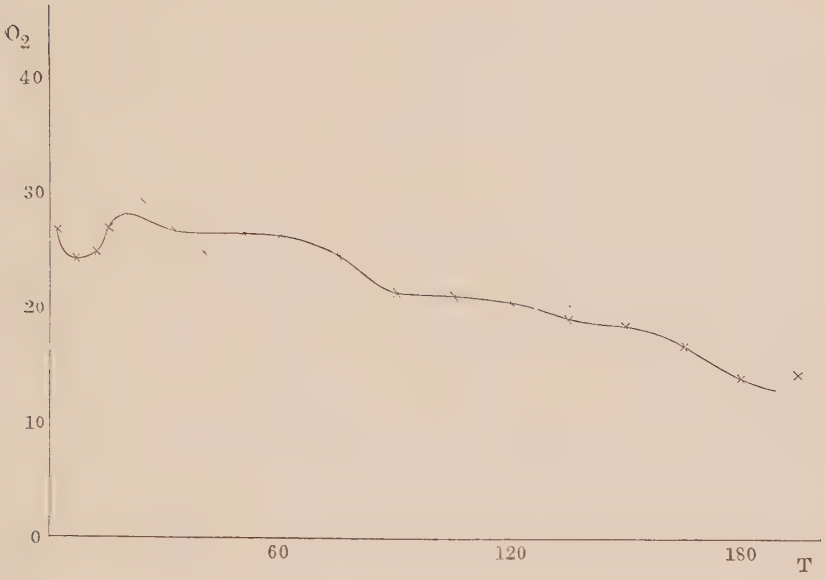


19. 0·5 gr. of cobalt chromate was added to the anode solution

Oxidation efficiency: 67·74% ($3^{h.}$) 65·80% ($3^{h.}$ $15^{m.}$)
Current efficiency: 56·71% 55·89%
Valeric acid formed: 6·92 gr.

Table XIII.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	26·7	40	24·7	120	20·5
7	24·2	50	26·4	135	19·1
12	24·8	60	26·3	150	18·5
17	26·9	75	24·6	165	16·8
24	29·3	90	21·3	180	14·0
32	26·6	105	21·0	195	14·4

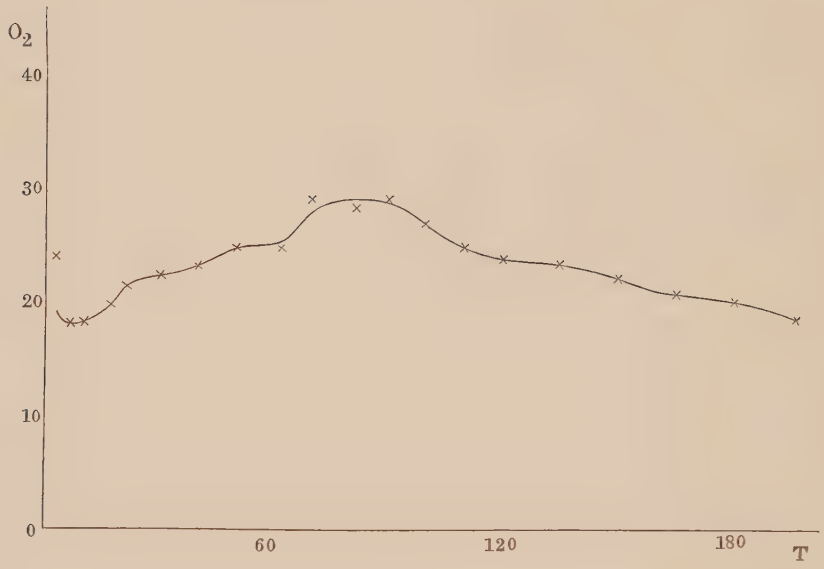


20. 0·5 gr. of uranium sulphate was added to the anode solution.

Oxidation efficiency: 72·57% ($3^{h.}$) 71·52% ($3^{h.}$ $15^{m.}$)
Current efficiency: 56·71% 55·89%
Valeric acid formed: 6·92 gr.

Table XIV.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	24·0	50	24·9	120	23·6
6	18·0	62	24·8	135	23·1
10	18·2	70	29·0	150	21·9
17	19·8	82	28·3	165	20·4
21	21·4	90	28·8	180	19·7
30	22·4	100	26·9	195	18·1
40	23·1	110	24·6		

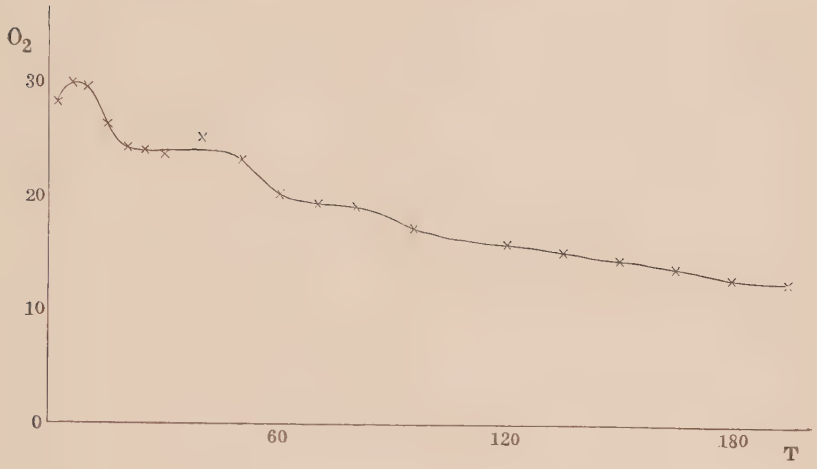


21. 1 gr. of manganous sulphate was used.

Oxidation efficiency: 58·19% ($3^{h.}$) 56·67% ($3^{h.}$ $15^{m.}$)
Current efficiency: 37·75% 36·77%
Valeric acid formed: 4·55 gr.

Table XV.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	28·3	40	25·2	135	15·3
6	29·9	50	23·1	150	14·5
10	29·6	60	20·1	165	13·8
15	26·3	70	19·4	180	12·8
20	24·1	80	19·1	195	12·6
25	24·0	95	17·2		
30	23 7	120	15 9		



22. 0·5 gr. of ferrous sulphate was used.

Oxidation efficiency: 92·02% (3^h.) 90·20% (3^h. 15^m.)
Current efficiency: 59·14% 57·97%
Valeric acid formed: 7·17 gr.

Table XVI.

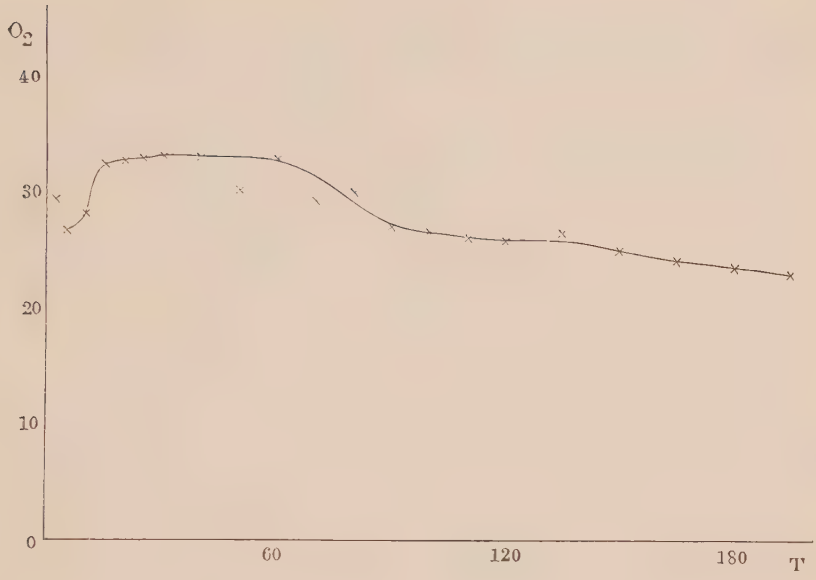
Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	32·1	40	33·3	110	27 0
5	32·8	50	33·3	120	29 7
10	32·9	60	33·1	135	29·3
15	33·1	70	32·9	150	26·3
20	32·7	80	32 1	165	23·9
25	33·0	90	32·5	180	23·5
30	33·0	100	30·2	195	21·3

23. 0·5 gr. of potassium bichromate was used.

Oxidation efficiency: 85·96% ($3^{h.}$) 84·84% ($3^{h.}$ $15^{m.}$)
Current efficiency: 60·81% 60·02%
Valeric acid formed: 7·43 gr.

Table XVII.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	29·5	40	33·3	110	26·4
5	26·7	50	30·4	120	26·2
10	28·1	60	33·2	135	27·0
15	32·5	70	29·6	150	25·4
20	32·7	80	30·2	165	24·5
25	33·0	90	27·4	180	23·8
30	33·3	100	27·0	195	23·2



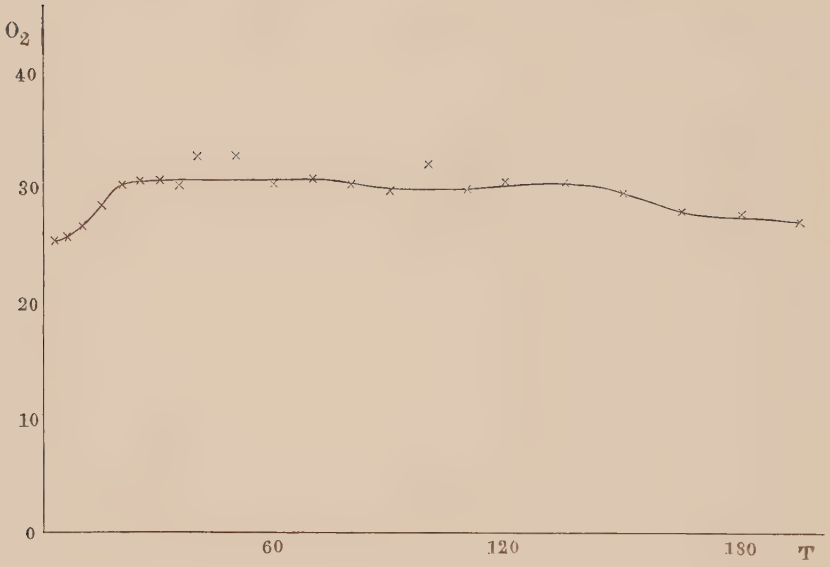
24. Anode: lead plate (216 sq. c.m.)

The other conditions being the same as in the previous experiment.

Oxidation efficiency: 90·10% ($3^{h.}$) 89·48% ($3^{h.}$ $15^{m.}$)
Current efficiency: 51·65% 5·130%
Valeric acid formed: 6·35 gr.

Table XVIII.

Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)	Time (minutes)	Oxygen (c.c.)
2	25·5	40	32·9	120	30·3
6	25·8	50	32·9	135	30·2
10	26·8	60	30·2	150	29·4
15	28·6	70	30·9	165	27·8
20	30·3	80	30·2	180	27·3
25	30·6	90	29·7	195	26·8
30	30·6	100	32·0		
35	30·2	110	29·8		



III. Electrolysis without diaphragm

25. Anode: a cylindrical vessel made of lead (periphery = 38·5 cm.; area of bottom = 117 sq. cm.) Area of anode (in contact with the electrolytic solution) = 425 sq. cm.

Electrolytic solution: 1000 c.c. H_2SO_4 (10%), 50 gr. isoamylalcohol.

Cathode: nickel wire about 1·5 m.m. dia.

Temperature: 22–24°

Current density: 1·412 amp./sq. dm.

Electric quantity: 6 amp. for $8^{h.}$ $48^{m.}$ (52·8 amp./h.)

Fig. 4.

In this experiment a glass stirrer as is shown in figure 4 was used with an excellent result.

Current efficiency: 56·70%

Valeric acid formed: 28·5 gr.

Comparing the above current efficiency with that obtained in the previous experiment conducted with a diaphragm—Vid. Table II.,—there is apparently no difference.

26. The conditions were the same as in exp. 25 except that 10 gr. of potassium bichromate were added to the electrolytic solution, and the current density was increased.

Current density:	1·88 amp./sq. dm.
Electric quantity:	8 amp. for $6^{h.}$ $36^{m.}$ (52·8 amp./h.)= 86·76% of the theoretical.
Current efficiency:	71·63%
Valeric acid formed:	36 gr.

The result of this experiment shows that the addition of $K_2Cr_2O_7$ greatly increases the current efficiency.

27. The conditions were the same as in exp. 26 except that potassium chlorate was used instead of potassium bichromate.

Current strength:	8 amp.
Time:	$7^{h.}$ $39^{m.}$
Current efficiency:	61·28%.
Valeric acid formed:	35 gr.

During electrolysis the electrode was greatly corroded.

SUMMARY.

1. Isovaleric acid, isovaleric aldehyde and carbon dioxide were observed to be produced by the electrolytic oxidation of isoamyl-alcohol.

2. Lead peroxide acted most effectively as anode material.

3. In an alkaline solution the acid formed was always accompanied by the aldehyde, and the more the current efficiency increased the smaller was the current density.

In the sulphuric acid solution aldehyde was produced only when current density less than 1 amp. to 100 sq. cm. was used.

For the formation of valeric acid the most favourable concentration of sulphuric acid was found to be 10–20%.

4. For the electrolytic preparation of valeric acid the most suitable current density was determined to be 1–1·4 amp./sq. dm. if the solution kept stationary, and about 2 amp./sq. dm. when it was stirred.

5. The temperature should be kept below 30°.

6. A convenient rapid method of electrolysis for preparing the valeric acid without a diaphragm was obtained.

7. As the oxygen carrier the following substances were found effective: Vanadium pentoxide; Cerium sulphate; Potassium bichromate; Ferrous sulphate.

In conclusion, the author wishes to take this opportunity of expressing his gratitude to Professor M. Matsui for his kind interest and valuable advice throughout the progress of this work.

Oriented Circles in Non-Euclidean Space II.

By

Hidetoshi Kashiwagi.

(Received March 17, 1922.)

Introduction.

In the present paper we want to discuss some properties about the circles in non-euclidean space, other than those which were not touched in a previous memoir.[1] In that paper Study's coordinates[2] were used. To study the incidence and the orthogonality of the circles, the coordinates of circles in analogy to those of Plücker's[3] in euclidean space may be used. Since the metric properties of the circles must be expressed by the relation between the radii of the circles, the coordinates of the centers and those of the planse of the circles, in the present study, I think it convenient to apply Study's coordinates of the circles in non-euclidean space.

I.

First, we set up our coordinates system as follows:

$$x_0 = k\cos\frac{\rho}{k},$$

$$x_1 = k\sin\frac{\rho}{k}\cos\alpha,$$

1, 2 T. Nishiuchi and H. Kashiwagi, Oriented circles in Non-Euclidean Space, Mem. Coll. Sci., Kyoto, **4**, No. 6.

3 The first writer to use these seems to have been Stephanos, 'Sur une configuration remarquable de cercles dans l'espace,' **C. R., 93,** 1881. See also Coolidge, 'Study of the circle cross,' Transac. Amer. Math. Soc., **14**.

$$x_2 = k \sin\frac{\rho}{k}\cos\beta,$$

$$x_3 = k \sin\frac{\rho}{k}\cos\gamma,$$

$$(xx) = x_0^2 + x_1^2 + x_2^2 + x_3^2 = k^2$$

with usual notations.[1]

A sphere in non-euclidean space may be given by the equations

$$(ax)^2 = k^4 \cos^2\frac{r}{k},$$

$$(aa) = (xx) = k^2$$

where (a) is the coordinates of the center of the sphere, r the radius of the sphere and $\frac{1}{k^2}$ the measure of curvature of the space.

According to the idea by which Sommerville[2] has treated the circle in non-euclidean plane, it is natural, for the equation of the sphere, to adopt the following linear one:

$$(ax) = k^2 \cos\frac{r}{k}.$$

Moreover I think it evident to take

$$\frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\cos\frac{r}{k}$$

as the coordinates of the sphere, where we take $\frac{a_0}{k} > o$ but $\frac{a_1}{k} > o$ provided $a_0 = o$ and so on. Let us adopt the following notation:

$$\frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\cos\frac{r}{k}$$

$$\equiv x_0,\ x_1,\ x_2,\ x_3,\ x_4$$

$$\equiv (x)$$

Let $(\bar{x})$ be the coordinate of the polar sphere of a sphere whose coordinates are (x), then

$$(\bar{x}) = \frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\sin\frac{r}{k}$$

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1 Coolidge, Non-Euclidean Geometry.

2 Non-Euclidean Geometry.

and

$$(xx)+(\bar{x}\bar{x})=1$$

And the coordinates of the null-sphere satisfy the following condition:

$$(xx)=0.$$

If θ be the angle of intersection of the two spheres whose coordinates are (x) and (y) respectively, then

$$\cos\theta=\frac{(xy)}{\sqrt{(xx)}\sqrt{(yy)}}.$$

The condition of orthogonal intersection will be

$$(xy)=0.$$

Next, if we put

$$i\sqrt{x_0^2+x_1^2+x_2^2+x_3^2+x_4^2}=x_5$$

then x_0, x_1, x_2, x_3, x_4, x_5 will also be the another coordinates of the sphere and

$$x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0.$$

Thus the condition of contact will be

$$x_0y_0+x_1y_1+x_2y_2+x_3y_3+x_4y_4+x_5y_5=0,$$

Let (x) and (y) be the coordinates of two spheres then the coordinates of the spheres which pass through the circles of intersection of these spheres will be

$$\lambda(x)+\mu(y)$$

where λ and μ are parameters.

II.

We may consider a circle as the intersection of two spheres whose centers are mutually orthogonal and one of whose radius is $\frac{\pi}{2}k$, i.e. the intersection of a sphere and a plane passing through the center of the sphere.

Then the determinants of the matrix

$$\left\|\begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{matrix}\right\|$$

where

$$x_0y_0+x_1y_1+x_2y_2+x_3y_3=o$$

$$y_4=o$$

define the Study's coordinates of the circle.

If

$$(x)=\frac{a_0}{k},\ \frac{a_1}{k},\ \frac{a_2}{k},\ \frac{a_3}{k},\ i\cos\frac{r}{k}$$

$$(y)=\frac{b_0}{k},\ \frac{b_1}{k},\ \frac{b_2}{k},\ \frac{b_3}{k},\ o,$$

then the Study's coordinates will be

$$\left\|\begin{matrix}\frac{a_0}{k} & \frac{a_1}{k} & \frac{a_2}{k} & \frac{a_3}{k} & i\cos\frac{r}{k}\\ \frac{b_0}{k} & \frac{b_1}{k} & \frac{b_2}{k} & \frac{b_3}{k} & o\end{matrix}\right\|$$

where

$$(ab)=o.$$

Hence, if we take $(\mathfrak{X}_0,\ \mathfrak{X}_1,\ \mathfrak{X}_2,\ \mathfrak{X}_3,\ \mathfrak{X}_{01},\ \mathfrak{X}_{02},\ \mathfrak{X}_{03},\ \mathfrak{X}_{23},\ \mathfrak{X}_{31},\ \mathfrak{X}_{12})\equiv(\mathfrak{X})$ be the Study's coordinates of the circle, then[1]

$$\mathfrak{X}_0\equiv i\frac{b_0}{k}\cos\frac{r}{k},\quad \mathfrak{X}_1\equiv i\frac{b_1}{k}\cos\frac{r}{k},\quad \mathfrak{X}_2\equiv i\frac{b_2}{k}\cos\frac{r}{k},\quad \mathfrak{X}_3\equiv i\frac{b_3}{k}\cos\frac{r}{k},$$

$$\mathfrak{X}_{01}\equiv\begin{vmatrix}\frac{a_0}{k} & \frac{a_1}{k}\\ \frac{b_0}{k} & \frac{b_1}{k}\end{vmatrix},\quad \mathfrak{X}_{02}\equiv\begin{vmatrix}\frac{a_0}{k} & \frac{a_2}{k}\\ \frac{b_0}{k} & \frac{b_2}{k}\end{vmatrix},\quad \mathfrak{X}_{03}\equiv\begin{vmatrix}\frac{a_0}{k} & \frac{a_3}{k}\\ \frac{b_0}{k} & \frac{b_3}{k}\end{vmatrix},$$

$$\mathfrak{X}_{23}\equiv\begin{vmatrix}\frac{a_2}{k} & \frac{a_3}{k}\\ \frac{b_2}{k} & \frac{b_3}{k}\end{vmatrix},\quad \mathfrak{X}_{31}\equiv\begin{vmatrix}\frac{a_3}{k} & \frac{a_1}{k}\\ \frac{b_3}{k} & \frac{b_1}{k}\end{vmatrix},\quad \mathfrak{X}_{12}\equiv\begin{vmatrix}\frac{a_1}{k} & \frac{a_2}{k}\\ \frac{b_1}{k} & \frac{b_2}{k}\end{vmatrix}.$$

And it is evident that there exists the following relations:

$$\mathfrak{X}_{ij}=-\mathfrak{X}_{ji}\qquad (i,\ j=0,\ 1,\ 2,\ 3)$$

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[1] T. Nishiuchi, loc. cit., p. 293.

$$\Omega_0(\mathfrak{X}_k\mathfrak{X}_{ij})\equiv\mathfrak{X}_1\mathfrak{X}_{23}+\mathfrak{X}_2\mathfrak{X}_{31}+\mathfrak{X}_3\mathfrak{X}_{12}=0,$$
$$\Omega_1(\mathfrak{X}_k\mathfrak{X}_{ij})\equiv\mathfrak{X}_2\mathfrak{X}_{30}+\mathfrak{X}_3\mathfrak{X}_{02}+\mathfrak{X}_0\mathfrak{X}_{23}=0,$$
$$\Omega_2(\mathfrak{X}_k\mathfrak{X}_{ij})\equiv\mathfrak{X}_3\mathfrak{X}_{01}+\mathfrak{X}_0\mathfrak{X}_{13}+\mathfrak{X}_1\mathfrak{X}_{30}=0,$$
$$\Omega_3(\mathfrak{X}_k\mathfrak{X}_{ij})\equiv\mathfrak{X}_0\mathfrak{X}_{12}+\mathfrak{X}_1\mathfrak{X}_{20}+\mathfrak{X}_2\mathfrak{X}_{01}=0,$$
$$(\mathfrak{X}_{ij}\mid\mathfrak{X}_{ij})\equiv\mathfrak{X}_{01}\mathfrak{X}_{23}+\mathfrak{X}_{02}\mathfrak{X}_{31}+\mathfrak{X}_{03}\mathfrak{X}_{12}=0.$$

It is easily seen that $\mathfrak{X}_{ij}$ $(i, j=0, 1, 2, 3)$ represent the axial coordinates of the plane-axis[1] of the circle and also represent the radial coordinates of the space axis.[2]

III.

Suppose we have two circles Γ and Γ' with Study's coordinates $(\mathfrak{X})$ and $(\mathfrak{X}')$, then their mutual power[3] $P(\Gamma, \Gamma')$ will be

$$\begin{aligned}P(\Gamma\Gamma')&=(aa')(bb')-(ab')(a'b)-k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb')\\&=k^4[\Sigma\mathfrak{X}_i\mathfrak{X}'_i+\Sigma\mathfrak{X}_{ij}\mathfrak{X}'_{ij}]\\&\equiv k^4[\mathfrak{X}\mathfrak{X}']\end{aligned}$$

and the condition that two non-coplanar circles should be cospherical is that

$$(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})=0,$$
$$\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})=\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})$$
$$(l=0, 1, 2, 3).$$

The coordinates of the points of the intersection of two non-coplanar and non-cospherical circles are given by

$$\Omega_0(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_0(\mathfrak{X}'_k\mathfrak{X}_{ij}):\Omega_1(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_1(\mathfrak{X}'_k\mathfrak{X}_{ij}):$$
$$:\Omega_2(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_2(\mathfrak{X}'_k\mathfrak{X}_{ij}):\Omega_3(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_3(\mathfrak{X}'_k\mathfrak{X}_{ij}).$$

And the conditions that two circles intersect at two points will be

$$(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})=0$$
$$\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})=0$$
$$(l=0, 1, 2, 3).$$

These are the conditions that the two circles should be cospherical. The coordinates of the center of the common orthagonal sphere

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1, 2, 3 T. Nishiuchi, loc. cit.

of the two non-coplanar and non-cospherical circles are also

$$\lambda[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]$$

where λ is proportional factor and

$$l=0,\ 1,\ 2,\ 3.$$

Now let R be the radius of common orthogonal sphere, then

$$\cos\frac{R}{k}=\frac{\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a'_0 & a'_1 & a'_2 & a'_3 \\ b'_0 & b'_1 & b'_2 & b'_3 \end{vmatrix}}{k\sqrt{\begin{Vmatrix} a_0\cos\frac{r'}{k}-a'_0\cos\frac{r}{k}\cdots\cdots a_3\cos\frac{r'}{k}-a'_3\cos\frac{r}{k} \\ b_0 \qquad\qquad\cdots\cdots\qquad\qquad b_3 \\ b'_0 \qquad\qquad\qquad\qquad b'_3 \end{Vmatrix}^2}}$$

$$=\frac{(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})}{\sqrt{\sum[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]^2}}$$

and

$$\tan^2\frac{R}{k}=\frac{\sum[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]^2-(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2}{(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2}.$$

If we put

$$\sum[\Omega_l(\mathfrak{X}_k\mathfrak{X}'_{ij})-\Omega_l(\mathfrak{X}'_k\mathfrak{X}_{ij})]^2-(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2\equiv(\mathfrak{X}\mid\mathfrak{X}')^2$$

then

$$\tan^2\frac{R}{k}=\frac{(\mathfrak{X}\mid\mathfrak{X}'^2)}{(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2}.$$

The condition that two non-coplaner and non-cospherical circles should interest is that

$$(\mathfrak{X}\mid\mathfrak{X}')^2=0.$$

The condition that two circles should be in involution is

$$[\mathfrak{X}\mathfrak{X}']=\sum\mathfrak{X}_i\mathfrak{X}'_i+\sum\mathfrak{X}_{ij}\mathfrak{X}'_{ij}=0.$$

For bi-involution we shall have

$$(\mathfrak{X}_i\mathfrak{X}'_i)\equiv\sum\mathfrak{X}_i\mathfrak{X}'_i=0,$$

$$\sum \mathfrak{A}_{ij} \begin{vmatrix} \mathfrak{A}_i \mathfrak{A}_j \\ \mathfrak{A}'\, \mathfrak{A}'_j \end{vmatrix} = \sum \mathfrak{A}'_{ij} \begin{vmatrix} \mathfrak{A}_i \mathfrak{A}_j \\ \mathfrak{A}'_i \mathfrak{A}'_j \end{vmatrix} = 0$$

and

$$(\mathfrak{A}_{ij} \mid \mathfrak{A}'_{ij}) + (\mathfrak{A}_i \mathfrak{A}_i)(\mathfrak{A}'_i \mathfrak{A}'_i) - (\mathfrak{A}_{ij} \mathfrak{A}_{ij})(\mathfrak{A}'_{ij} \mathfrak{A}'_{ij}) = 0.$$

Let

$$\begin{Vmatrix} \frac{a_0}{k} & \frac{a_1}{k} & \frac{a_2}{k} & \frac{a_3}{k} & i\cos\frac{r}{k} \\ \frac{b_0}{k} & \frac{b_1}{k} & \frac{b_2}{k} & \frac{b_3}{k} & 0 \end{Vmatrix}$$

are the coordinates of a circle. Then the coordinates of the spheres passing through the circle are

$$\lambda\frac{a_0}{k}+\mu\frac{b_0}{k},\ \lambda\frac{a_1}{k}+\mu\frac{a_1}{k},\ \lambda\frac{a_2}{k}+\mu\frac{b_2}{k},\ \lambda\frac{a_3}{k}+\mu\frac{b_3}{k},\ \lambda i\cos\frac{r}{k},$$

where λ and μ are proportional factors.

Hence the coordinates of the foci of the circle (i.e. the null-sphere passing through the circle) are

$$(x)\equiv \frac{a_0}{k}+i\frac{b_0}{k}\sin\frac{r}{k},\ \frac{a_1}{k}+i\frac{b_1}{k}\sin\frac{r}{k},\ \ldots\ldots,\ \frac{a_3}{k}+i\frac{b_3}{k}\sin\frac{r}{k},\ i\cos\frac{r}{k},$$

$$(y)\equiv \frac{a_0}{k}-i\frac{b_0}{k}\sin\frac{r}{k},\ \frac{a_1}{k}-i\frac{b_1}{k}\sin\frac{r}{k},\ \ldots\ldots,\ \frac{a_3}{k}-\frac{b_3}{k}\sin\frac{r}{k},\ i\cos\frac{r}{k},$$

and that of the polar circles are

$$(\bar{x})\equiv \frac{a_0}{x}+i\frac{b_0}{k}\cos\frac{r}{k},\ \frac{a_1}{k}+i\frac{b_1}{k}\cos\frac{r}{k},\ \ldots\ldots,\ \frac{a_3}{k}+i\frac{b_3}{k}\cos\frac{r}{k},\ i\sin\frac{r}{k},$$

$$(\bar{y})\equiv \frac{a_0}{k}-i\frac{b_0}{k}\cos\frac{r}{k},\ \frac{a_1}{k}-i\frac{b_1}{k}\cos\frac{r}{k},\ \ldots\ldots,\ \frac{a_3}{k}-i\frac{b_3}{k}\cos\frac{r}{k},\ i\sin\frac{r}{k}.$$

When two circles have the same plane-axis (consequently they have the same space-axis), it is evident that

$$\mathfrak{A}_{ij} = \mathfrak{A}'_{ij}$$

and they intersect one another on the absolute.

IV.

Definition—Two cospherical circles are said to be orthogonal when they are in involution.

Next we shall call *common perpendicular* to two non-cospherical and non-coplaner circles, a circle cospherical and orthogonal to both.

Let (x), (y) be the coordinates of foci of the first circle and (x') (y') be those of the second circles, whose coordinates are

$$\begin{Vmatrix} \frac{a_0}{k} & \frac{a_1}{k} & \cdots\cdots & i\cos\frac{r}{k} \\ \frac{b_0}{k} & \frac{b_1}{k} & \cdots\cdots & 0 \end{Vmatrix}$$

and

$$\begin{Vmatrix} \frac{a'_0}{k} & \frac{a'_1}{k} & \cdots\cdots & i\cos\frac{r'}{k} \\ \frac{b'_0}{k} & \frac{b'_1}{k} & \cdots\cdots & 0 \end{Vmatrix}$$

respectively.

If such a common perpendicular be determined by the sphere $\lambda(x)+\mu(y)$, $\lambda'(x')+\mu'(y')$, there must be some sphere through each circle that is orthogonal to each of these spheres. (Since the common perpendicular and each of circles are in involution). This will require that $\lambda(x)-\mu(y)$ be orthogonal to $\lambda'(x')+\mu'(y')$ and $\lambda'(x')-\mu'(y')$ orthogonal to $\lambda(x)+\mu(y)$. Hence we have

$$\lambda\lambda'(xx')-\mu\mu'(yy')=0$$
$$\mu\lambda'(yx')-\lambda\mu'(xy')=0.$$

Eliminating $\frac{\lambda'}{\mu'}$ and $\frac{\lambda}{\mu}$,

$$\lambda^2(xx')(xy')-\mu^2(yx')(yy')=0$$
$$\lambda'^2(xx')(yx')-\mu'^2(xy')(yy')=0.$$

Hence, we have two pairs of spheres

$$\sqrt{(yx')}\sqrt{(yy')}(x)\pm\sqrt{(xx')}\sqrt{(xy')}(y),$$
$$\sqrt{(y'x)}\sqrt{(yy')}(x')\pm\sqrt{(x'x)}\sqrt{(x'y)}(y')$$

whose intersections gives the common perpendiculars of two given circles. And they are orthogonal to one another.

Theorem. There are two pairs of common perpendiculars to two non-cospherical circles, and two circles of different pairs are in bi-involution.

We shall define as the *angles of two circles* those of two pairs of spheres.

If θ_1 and θ_2 are the angles of those spheres, we find

$$\cos\theta_1^2\cos\theta_2^2=\frac{\begin{vmatrix}(xx')&(xy')\\(yx')&(yy')\end{vmatrix}^2}{(xy)^2(x'y')^2}$$

$$=\frac{1}{(xy)^2(x'y')^2}\left\{\begin{Vmatrix}x_0&x_1&x_2&x_3&x_4\\y_0&y_1&y_2&y_3&y_4\end{Vmatrix}\cdot\begin{Vmatrix}x'_0&x'_1&x'_2&x'_3&x'_4\\y'_0&y'_1&y'_2&y'_3&y'_4\end{Vmatrix}\right\}$$

$$=\frac{1}{\sin^4\frac{r}{k}\sin^4\frac{r'}{k}}\left\{\begin{Vmatrix}\frac{a_0}{k}&\cdots\cdots&\frac{a_3}{k}&i\cos\frac{r}{k}\\ i\frac{b_0}{k}\sin\frac{r}{k}&\cdots\cdots i\frac{b_3}{k}\sin\frac{r}{k}&0\end{Vmatrix}\times\right.$$

$$\left.\times\begin{vmatrix}\frac{a'_0}{k}&\cdots\cdots&i\cos\frac{r'}{k}\\ i\frac{b'_0}{k}\sin\frac{r'}{k}&\cdots\cdots&0\end{vmatrix}\right\}^2$$

$$=\frac{1}{\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}\begin{vmatrix}\frac{(aa')}{k^2}-\cos\frac{r}{k}\cos\frac{r'}{k}&i\frac{(ab')}{k^2}\\ i\frac{(a'b)}{k^2}&-\frac{(bb')}{k^2}\end{vmatrix}^2$$

$$=\frac{\left[(aa')(bb')-(ab')(a'b)-k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb')\right]^2}{k^8\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}$$

$$=\frac{P(\Gamma,\Gamma')^2}{k^8\sin^2\frac{r}{k}\sin^2\frac{r'}{k}}$$

$$=\frac{[\mathfrak{X}\mathfrak{X}']^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}.$$

$$\cos^2\theta_1+\cos^2\theta_2=\frac{2[(xx')(yy')+(yx')(xy')]}{(xy)(x'y')}.$$

$$\sin^2\theta_1\sin^2\theta_2=1-(\cos^2\theta_1+\cos^2\theta_2)+\cos^2\theta_1\cos^2\theta_1$$

$$=\frac{\begin{Vmatrix}x_0&x_1&x_2&x_3&x_4\\y_0&y_1&y_2&y_3&y_4\\x'_0&x'_1&x'_2&x'_3&x'_4\\y'_0&y'_1&y'_2&y'_3&y'_4\end{Vmatrix}^2}{(xy)^2(x'y')^2}$$

$$= \frac{1}{\sin^2\frac{r}{k}\sin^2\frac{r'}{k}} \left\| \begin{matrix} \frac{a_0}{k} & \frac{a_1}{k} & \cdots\cdots & \frac{a_3}{k} & i\cos\frac{r}{k} \\ \frac{b_0}{k} & \frac{b_1}{k} & \cdots\cdots & \frac{b_3}{k} & 0 \\ \frac{a'_0}{k} & \frac{a'_1}{k} & \cdots\cdots & \frac{a'_3}{k} & i\cos\frac{r'}{k} \\ \frac{b'_0}{k} & \frac{b'_1}{k} & \cdots\cdots & \frac{b'_3}{k} & 0 \end{matrix} \right\|^2$$

$$= \frac{1}{\sin^2\frac{r}{k}\sin^2\frac{r'}{k}} \left[\left| \begin{matrix} \frac{a}{k} & \frac{a'}{k} & \frac{b}{k} & \frac{b'}{k} \end{matrix} \right|^2 + \left\| \begin{matrix} \frac{a_0}{k}\cos\frac{r'}{k} - \frac{a'_0}{k}\cos\frac{r}{k} & \cdots \\ \frac{b_0}{k} & \cdots\cdots \\ \frac{b'_0}{k} & \cdots\cdots \end{matrix} \right\|^2 \right]$$

$$= \frac{(\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\cos^2\theta_1 + \cos^2\theta_2 = \frac{[\mathfrak{X}'\mathfrak{X}]^2 - (\mathfrak{X} \mid \mathfrak{X}')^2 + [\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\theta_1 + \sin^2\theta_2 = \frac{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}'] - [\mathfrak{X}\mathfrak{X}']^2 + (\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

V.

Here, we shall discuss only about the oriented circles.

It is evident that two oriented circles have two common perpendiculars. Let θ_1 and θ_2 be their angles then we have

$$\cos^2\theta_1 \cos^2\theta_2 = \frac{[\mathfrak{X}\mathfrak{X}']^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']},$$

$$\sin^2\theta_1 \sin^2\theta_2 = \frac{(\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']},$$

$$\cos^2\theta_1 + \cos^2\theta_2 = \frac{[\mathfrak{X}\mathfrak{X}']^2 - (\mathfrak{X} \mid \mathfrak{X}')^2 + [\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\theta_1 + \sin^2\theta_2 = \frac{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}'] - [\mathfrak{X}\mathfrak{X}']^2 + (\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']}$$

Hence, $\cos\theta_1$, $\cos\theta_2$ and $\sin\theta_1$, $\sin\theta_2$ are the roots of the equations.

$$[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']\cos^4\theta+\Big[(\mathfrak{A}\mid\mathfrak{A}')^2-[\mathfrak{A}\mathfrak{A}']^2-[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']\Big]\cos^2\theta+[\mathfrak{A}\mathfrak{A}']^2=0$$

$$[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']\sin^4\theta+\Big[[\mathfrak{A}\mathfrak{A}']^2-(\mathfrak{A}\mid\mathfrak{A}')^2-[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']\Big]\sin^2\theta+(\mathfrak{A}\mid\mathfrak{A}')^2=0.$$

The condition that these angles should be equal or supplementary is

$$\Big[(\mathfrak{A}\mid\mathfrak{A}')^2-[\mathfrak{A}\mathfrak{A}']^2-[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']\Big]^2-4[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}'][\mathfrak{A}\mathfrak{A}']^2=0.$$

or

$$\Big\{\Big[(\mathfrak{A}\mid\mathfrak{A}')+[\mathfrak{A}\mathfrak{A}']\Big]^2-[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']\Big\}\Big\{\Big[(\mathfrak{A}\mid\mathfrak{A}')-[\mathfrak{A}\mathfrak{A}']\Big]^2-[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']\Big\}=0.$$

When two circles intersect on the absolute and touch one another, we shall call these circles are *parallel.*

These conditions will be expressed by the equations

$$\left|\begin{array}{cccc} a & a' & b & b' \end{array}\right|^2=\left\|\begin{array}{cccc} a\cos\dfrac{r'}{k} & -a'\cos\dfrac{r}{k} & b & b' \end{array}\right\|^2$$

$$=k^8\sin^2\frac{r}{k}\sin^2\frac{r'}{k}-\Big(P(\Gamma\Gamma')\Big)^2=0$$

or

$$(\mathfrak{A}\mid\mathfrak{A}')^2=(\mathfrak{A}_{ij}\mid\mathfrak{A}'_{ij})=[\mathfrak{A}\mathfrak{A}][\mathfrak{A}'\mathfrak{A}']-[\mathfrak{A}\mathfrak{A}']^2=0.$$

When two circles are parallel then their space axes and also plane axes intersect to each other and their points of intersection is on the absolute. Hence we have the theorem.

Theorem. When two circles are parallel their space-axis and the plane axes are respectively either parallel or psodoparallel.

From this theorem the condition of parallelism will be

$$\left| \begin{matrix} a & a' \\ b & b' \end{matrix} \right|^2 = \left\| \begin{matrix} a\cos\frac{r'}{k} - a' & \cos\frac{r}{k} \\ b & b' \end{matrix} \right\|^2$$

$$= k^8 \sin^2\frac{r}{k}\sin^2\frac{r'}{k} - \Big(P(\Gamma\Gamma')\Big)^2$$

$$k^8 - \Big[(aa')(bb') - (ab')(a'b)\Big]^2 = 0.$$

or

$$(\mathfrak{X} \mid \mathfrak{X}')^2 = (\mathfrak{X}_{ij} \mid \mathfrak{X}'_{ij})$$

$$= [\mathfrak{X}\,\mathfrak{X}]\,[\mathfrak{X}'\mathfrak{X}'] - [\mathfrak{X}\,\mathfrak{X}']^2$$

$$= (\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij}) - (\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2 = 0.$$

From these conditions we have

$$\left[\pm k^4 - k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb')\right]^2 = k^8\sin\frac{r}{k}\sin^2\frac{r'}{k}$$

or

$$\left[\pm k^4 - k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb') - k^4\sin\frac{r}{k}\sin\frac{r'}{k}\right]$$

$$\left[\pm k^4 - k^2\cos\frac{r}{k}\cos\frac{r'}{k}(bb') + k^4\sin\frac{r}{k}\sin\frac{r'}{k}\right] = 0.$$

Hence we have the theorem:

Theorem. When two circles are parallel, if their radii is equal to zero or differ by πk, then the planes of circles are parallel or two circles are coplanar and vice versa.

d_1, d_2 which satisfy the following equations are called the *distances of two oriented circles* whose coordinate are $(\mathfrak{X})$, $(\mathfrak{X}')$

$$\cos^2\frac{d_1}{k}\cos^2\frac{d_2}{k} = \frac{[\mathfrak{X}\,\mathfrak{X}']}{[\mathfrak{X}\,\mathfrak{X}]\,[\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\frac{d_1}{k}\sin^2\frac{d_2}{k} = \frac{(\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\,\mathfrak{X}]\,[\mathfrak{X}'\mathfrak{X}']}$$

$$\cos^2\frac{d_1}{k} + \cos^2\frac{d_2}{k} = \frac{[\mathfrak{X}\,\mathfrak{X}'] - (\mathfrak{X} \mid \mathfrak{X}')^2 + [\mathfrak{X}\,\mathfrak{X}]\,[\mathfrak{X}'\mathfrak{X}']}{[\mathfrak{X}\,\mathfrak{X}]\,[\mathfrak{X}'\mathfrak{X}']}$$

$$\sin^2\frac{d_1}{k} + \sin^2\frac{d_2}{k} = \frac{[\mathfrak{X}\,\mathfrak{X}]\,[\mathfrak{X}'\mathfrak{X}'] - [\mathfrak{X}\,\mathfrak{X}']^2 + (\mathfrak{X} \mid \mathfrak{X}')^2}{[\mathfrak{X}\,\mathfrak{X}]\,[\mathfrak{X}'\mathfrak{X}']}$$

We thus get the following equations for the distances d of the two circles $(\mathfrak{X})$, $(\mathfrak{X}')$

$$(A)\quad \begin{aligned} &[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\cos^4\frac{d}{k} \\ &\quad+\left[(\mathfrak{X}\mid\mathfrak{X}')^2-[\mathfrak{X}\mathfrak{X}']^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\right]\cos^2\frac{d}{k}+[\mathfrak{X}\mathfrak{X}']^2=0, \\ &[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\sin^4\frac{d}{k} \\ &\quad+\left[[\mathfrak{X}\mathfrak{X}']^2-(\mathfrak{X}\mid\mathfrak{X}')^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\right]\sin^2\frac{d}{k}+(\mathfrak{X}\mid\mathfrak{X}')^2=0. \end{aligned}$$

The square roots of the products of the roots of those two equations shall be called *commoment*[1] and *moment*[2] of two circles respectively.

If two circles intersect the moment must be zero and if their mutual powers are equal to zero the commoment must be vanished.

Two circles shall be said to be *paratactic*[3] when their distances and angles are all congruent.

If two circles be paratactic the condition may be expressed analytically by equating to zero the discriminant of either of our equations (*A*).

$$\left\{\left[(\mathfrak{X}\mid\mathfrak{X}')+[\mathfrak{X}\mathfrak{X}']\right]^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\right\} \times\left\{\left[(\mathfrak{X}\mid\mathfrak{X}')-[\mathfrak{X}\mathfrak{X}']\right]^2-[\mathfrak{X}\mathfrak{X}][\mathfrak{X}'\mathfrak{X}']\right\}=0.$$

This puts in evidence that intersecting circles cannot be paratatic unless they be parallel.

If

$$\left(a+ib\sin\frac{r}{k}\;\;a'+ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}$$

$$\left(a-ib\sin\frac{r}{k}\;\;a'-ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}$$

or

$$\left(a+ib\sin\frac{r}{k}\;\;a'-ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}$$

$$\left(a-ib\sin\frac{r}{k}\;\;a'+ib'\sin\frac{r'}{k}\right)=k^2\cos\frac{r}{k}\cos\frac{r'}{k}$$

1, 2, 3 This word is adopted in analogy to that of in Line Geometry. See D'Ovidio, 'Studio sulla geometria proiettiva,' Annali di Mathematica, **6**, 1873, and see Study, 'Zur Nicht-Euclidischen und Linien- Geometrie,' Jahresber. D. M. Ver., **11**, 1902.

i.e. the focus of one is orthogonal to that of the other circle, then the equation above will be satisfied. Hence we get the theorem.

THEOREM. *If a focus of one circle is orthogonal to the other, two circles are paratactic.*

If the radius of the both circles be $\frac{\pi}{2}k$, then the circles will be two straight lines whose Plücker's coordinates are $(\mathfrak{X}_{ij})$ and $(\mathfrak{X}'_{ij})$ respectively, and the equations (A) will be

$$(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\sin^4\frac{d}{k}$$
$$+\left[(\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2-(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2-(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\right]\sin^2\frac{d}{k}$$
$$+(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2=0.$$

$$(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\cos^4\frac{d}{k}$$
$$+\left[(\mathfrak{X}_{ij}\mid\mathfrak{X}_{ij}')^2-(\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2-(\mathfrak{X}_{ij}\mathfrak{X}_{ij})(\mathfrak{X}'_{ij}\mathfrak{X}'_{ij})\right]\cos^2\frac{d}{k}$$
$$+(\mathfrak{X}_{ij}\mathfrak{X}'_{ij})^2=0.$$

These are the equations which give the distances between the two lines $(\mathfrak{X}_{ij})$, $(\mathfrak{X}'_{ij})$[1].

VI.

Lastly, we shall express the pseudo-moment[2] and pseudo-commoment[3] of two circles by the study's coordinates. Let d_1' and d_2' be the pseudo distances of two circles then the square of the pseudo moment will be

$$\sin^2\frac{d_1'}{k}\sin^2\frac{d_2'}{k}=\frac{|\,a\;a'\;b\;b'\,|^2\tan^4\frac{R}{k}}{\left(\sin^2\frac{r}{k}+\cos^2\frac{r}{k}\tan^2\frac{r}{k}\right)\left(\sin^2\frac{r'}{k}+\cos^2\frac{r'}{k}\tan^2\frac{R}{K}\right)}$$

$$=\frac{k^4[(\mathfrak{X}\mid\mathfrak{X}')^2]^2}{\begin{vmatrix}(\mathfrak{X}_{ij}\mid\mathfrak{X}_{ij}')^2(\mathfrak{X}_i\mathfrak{X}_i) & (\mathfrak{X}_{ij}\mid\mathfrak{X}_{ij}')^2(\mathfrak{X}_i'\mathfrak{X}_i')\\ (\mathfrak{X}\mid\mathfrak{X}')^2\quad[\mathfrak{X}\mathfrak{X}] & (\mathfrak{X}\mid\mathfrak{X}')^2\quad[\mathfrak{X}'\mathfrak{X}']\end{vmatrix}}$$

And the square of their pseudo-commoment will be

---

1 Coolidge, Non-Euclidean Geometry.

2 3 T. Nischiuchi, loc. cit.

$$\cos^2\frac{d_1'}{k}\cos^2\frac{d_2'}{k}=\frac{\left\{P(\Gamma\Gamma')+(bb')\cos\frac{r}{k}\cos\frac{r'}{k}\tan\frac{R}{k}\right\}^2}{\left\{\sin^2\frac{r}{k}+\cos^2\frac{r}{k}\tan^2\frac{R}{K}\right\}\left\{\sin^2\frac{r'}{k}\cos^2\frac{r'}{k}\tan^2\frac{R}{K}\right\}}$$

$$=\frac{k^4\begin{vmatrix}(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2 & (\mathfrak{X}_i\mathfrak{X}'_i)\\ (\mathfrak{X}\mid\mathfrak{X}')^2 & [\mathfrak{X}\mathfrak{X}']\end{vmatrix}^2}{\begin{vmatrix}(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2 & (\mathfrak{X}_i\mathfrak{X}_i)\\ (\mathfrak{X}\mid\mathfrak{X}')^2 & [\mathfrak{X}\mathfrak{X}]\end{vmatrix}\cdot\begin{vmatrix}(\mathfrak{X}_{ij}\mid\mathfrak{X}'_{ij})^2 & (\mathfrak{X}_i'\mathfrak{X}_i')\\ (\mathfrak{X}\mid\mathfrak{X}')^2 & [\mathfrak{X}'\mathfrak{X}']\end{vmatrix}}$$

In conclusion, I wish to discuss, in the next paper, the systems of circles and their differential Geometry in non-euclidean space by the aid of Study's coordinates.

The author wishes to express his sincere thanks to Prof. T. Nishiuchi for his kind guidance and valuable remarks.

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Vol. V, No. 6, Published July 15th, 1922.

Price - - - Yen 1.00

These Memoirs are on sale at

MARUZEN COMPANY, LTD.

TOKYO, OSAKA, KYOTO, YOKOHAMA, NAGOYA, FUKUOKA & SENDAI.

大正十一年七月十三日印刷
大正十一年七月十五日發行

編纂兼發行者　京都帝國大學理學部

印刷者　大久保秀次郎
東京府下荏原郡世田谷村字下町五十番地

印刷所　株式會社 東京築地活版製造所
東京市京橋區築地二丁目十七番地

賣捌所　丸善株式會社
東京、大阪、京都、横濱、名古屋、福岡、仙臺

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